

Hodographic Vortices

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Abstract

Vortices are screw phase dislocations associated with helicoidal wave-fronts. In nonlinear optics, vortices arise as singular solutions to the phase-intensity equations of geometric optics. They exist for a general class of nonlinear response functions. In this sense, vortices possess a universal character. Analysis of geometric optics equations on the hodograph plane leads to deformed vortex type solutions that are sensitive to the form of the nonlinearity. The case of a Kerr type nonlinear response is discussed as a specific example.

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1 Introduction

Vortices are fundamental physical objects. They appear in many different contexts: from fluid mechanics to nonlinear optics, from superfluidity to Bose-Einstein condensation [1, 2, 3, 4]. In 1974, J.F. Nye and M. Berry introduced the concept of phase dislocation obtained by interference of quasi-monochromatic wave trains [5]. In optics, such phase singularities are called optical vortices and, as suggested by Couillet *et al.* [6], correspond to a state analog to vortices in superfluidity. Vortex solitons in Kerr media have been numerically predicted and experimentally observed by Swartzlander and Law in 1992 [7].

The discovery of vortex solitons immediately attracted great interest giving birth to a new branch on modern optics referred to as nonlinear singular optics [2]. Recent developments, in this field, concern the stabilization of optical vortices by propagation in nonlocal nonlinear media [8, 9].

In the linear theory, phase dislocations result from an interference phenomenon and, consequently, are not observable in the geometric optics limit. The situation is completely different in nonlinear regime where the geometric optics limit does not exclude the existence of phase dislocations.

In the present Letter, we are interested in the study of possible new singular phase solutions arising in nonlinear geometric optics. Phase dislocations associated with such solutions have a purely geometric nature and their properties are sensitive to the form of the nonlinear optical response.

We emphasize that the analysis of the long wave limit is also important for the study of the full dispersive regime. New singular phase solutions in nonlinear geometric optics can be used in the construction of a nontrivial ansatz for solving the nonlinear wave equation via, for instance, variational or numerical methods.

The Letter is organized as follows: in section 2, we derive the phase-intensity equations for a monochromatic light beam propagating in a generic nonlinear medium. We compute the standard vortex (SV) associated with a helicoidal wave-front and discuss its universal character. In section 3, a brief review of the hodograph method is presented. In section 4, we compute, using the hodograph method, a new family of singular phase solutions for a Kerr type nonlinear response. We call such solutions *hodographic vortices*. Hodographic vortices turn out to be a geometric deformation of the SVs.

2 Standard vortex

Let us consider a stationary paraxial light beam propagating through a weakly nonlinear medium of refractive index $n^2(I) = n_0^2 + \alpha^2 n_1^2(I)$, where $I = |\mathbf{E}|^2$ is the intensity of the electric field. We assume $n_1^2(I)$ to be a monotonic increasing function of the intensity, vanishing at $I = 0$. The constant n_0 is the value of the refractive index in absence of electric field. The weakly nonlinear regime is specified by the maximal value of the intensity I_{max} such that the nonlinearity acts as a small perturbation of the background refractive index. Such a perturbation is characterized by the small parameter $\alpha^2 = (n^2(I_{max}) - n_0^2)/n_0^2$ associated with the maximal variation of the refractive index induced by the electric field. Slow modulations of the electric field, propagating along the spatial direction z , are described by the following nonlinear Schrödinger (NLS) type equation

$$2ik \frac{\partial A}{\partial Z} + \frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} + k_0 n_1^2(|A|^2)A = 0, \quad (1)$$

where $A = A(X, Y, Z)$ is the envelope of a linearly polarized electric field $E = A \exp i(Z/\alpha^2 - \omega t)$, $k_0 = \omega/c$ ($c \equiv$ light speed) and $k = k_0 n_0$. For our purposes, it is convenient rescaling the equation (1) to the dimensionless form.

Let us s_0 be the typical spot-size, $L_d = ks_0^2$ the diffraction length and $L_{nl} = 1/(k_0 n_1^2(I_{max}))$ the nonlinear length. Introducing the dimensionless variables $x = X/(\sqrt{2}s_0)$, $y = Y/(\sqrt{2}s_0)$, $z = Z/(2\sqrt{L_{nl}L_d})$, $\psi = A/\sqrt{I_{max}}$, and the quantity $\eta = n_1^2 L_{nl}/n_0$, the NLS equation (1) takes the following standard form

$$i\epsilon\psi_z + \frac{\epsilon^2}{2}\nabla^2\psi + \eta(|\psi|^2)\psi = 0, \quad (2)$$

where $\epsilon = \sqrt{L_{nl}/L_d}$, the subscript denotes the partial differentiation and $\nabla = (\partial_x, \partial_y)$. Low dispersion/nonlinear geometric optics limit is obtained assuming the diffraction length L_d to be much larger than the nonlinear length L_{nl} , i.e. $\epsilon \ll 1$. We perform this limit in a standard fashion, looking for high oscillating solutions of the form $\psi = \phi \exp(iS/\epsilon)$. Introducing the slow variables $u = |\phi|^2$, $v = S_x$ and $w = S_y$ (note that by definition $v_y = w_x$) it is straightforward to show that, in the limit $\epsilon \rightarrow 0$, Eq. (2) is equivalent to the following dispersionless NLS type equation

$$u_z + (uv)_x + (uw)_y = 0 \quad (3a)$$

$$v_z + vv_x + ww_y - \eta_x = 0 \quad (3b)$$

$$w_z + ww_y + vv_x - \eta_y = 0. \quad (3c)$$

We refer to the monotonic function $\eta(u)$ as *intensity law*. We point out that, in paraxial approximation, Eq. (3a) is nothing but the Poynting vector conservation law and the equations (3b-3c) are equivalent to the eikonal equation.

Let us look for stationary solutions to the system of Eqs. (3) such that $S = z + F(x, y)$ and $u = u(x, y)$. Note that the functions $v = F_x$ and $w = F_y$ are the transverse components of the gradient vector $(v, w, 1)$ which is orthogonal to the wavefront $(x, y, F(x, y))$. In terms of the function $F(x, y)$, Eqs. (3) are equivalent to the following equations

$$F_x^2 + F_y^2 = 2\eta(u) \quad (4a)$$

$$u(F_{xx} + F_{yy}) + u_x F_x + u_y F_y = 0. \quad (4b)$$

This system of equations is known in fluid dynamics as a model for the two-dimensional steady, adiabatic irrotational compressible flow. Function F plays the role of the potential velocity and u the density of the fluid.

Looking at u as a function of η (this can always be done due the monotonicity of the intensity law), and using the Eq. (4a) into the Eq. (4b), we get the quasilinear equation of the form

$$AF_{xx} + BF_{yy} + 2CF_{xy} = 0, \quad (5)$$

where $A = JF_x^2 + 1$, $B = JF_y^2 + 1$, $C = JF_xF_y$ and $J = d(\log u(\eta))/d\eta$. The second order equation (5) is said to be elliptic if its discriminant $\Delta = AB - C^2 = 4J\eta + 1$ is strictly positive. The ellipticity condition $\Delta > 0$ is uniformly (i.e. for any solution) satisfied for few physically relevant intensity laws. Important examples are the focussing Kerr-type ($u = \eta^\gamma$) and logarithmic saturable ($\eta = \log(1 + u)$) nonlinear responses. For this reason, in the following, we restrict ourselves to the study of the elliptic case only.

It was observed in Ref. [10], that, if the function F is harmonic, the corresponding wave-front is a harmonic minimal surface. Indeed, provided F to satisfy the Laplace equation $F_{xx} + F_{yy} = 0$, then F solves the equation (5) for any J , and consequently for any function $u(\eta)$, if and only if it is a solution to the minimal surfaces equation

$$(1 + F_y^2)F_{xx} + (1 + F_x^2)F_{yy} - 2F_xF_yF_{xy} = 0. \quad (6)$$

Moreover, it can be proved that the only harmonic minimal surface of the form $(x, y, F(x, y))$ is the helicoid $(x, y, \arctan(x/y))$ [11]. This singular wave-front is the SV associated with screw type phase dislocations [5]. However, in nonlinear geometric optics, SVs have a purely geometric origin since they do not result from an interference phenomenon. We emphasize their ‘universal’ character due to the fact that SVs are solutions to the system of Eqs. (4) for an arbitrary monotonic intensity law.

In the following, we construct *hodographic vortices* as a family of vortex-type solutions to Eqs. (4). Unlike the SVs, hodographic vortices are not universal in the sense specified above. In fact, their geometric structure turns out to be depending on the specific form of the nonlinear response.

3 Hodograph Method

The system of Eqs. (4) can be linearized by a hodograph transformation. The hodograph method is widely used in fluid dynamics for the study of two dimensional compressible flows [12].

Let us introduce the *stream function* φ via equations

$$uv = \varphi_y, \quad uw = -\varphi_x, \quad (7)$$

where $v = F_x$ and $w = F_y$. Using Eqs. (7) into the identity $v_y = w_x$, one gets the following equation for φ :

$$u(\varphi_{xx} + \varphi_{yy}) - u_x\varphi_x - u_y\varphi_y = 0. \quad (8)$$

Let us introduce polar coordinates $v = p \cos \theta$, $w = p \sin \theta$ and suppose that the hodograph transformation $x = x(p, \theta)$, $y = y(p, \theta)$ exists. Expanding the total differentials $dx(p, \theta)$, $dy(p, \theta)$ and $dF(x(p, \theta), y(p, \theta))$, $d\varphi(x(p, \theta), y(p, \theta))$, we obtain the following set of equations

$$\begin{aligned} x_p &= \frac{\cos \theta}{p} F_p - \frac{\sin \theta}{up} \varphi_p, & x_\theta &= \frac{\cos \theta}{p} F_\theta - \frac{\sin \theta}{up} \varphi_\theta, \\ y_p &= \frac{\sin \theta}{p} F_p + \frac{\cos \theta}{up} \varphi_p, & y_\theta &= \frac{\sin \theta}{p} F_\theta + \frac{\cos \theta}{up} \varphi_\theta. \end{aligned} \quad (9)$$

The system of Eqs. (9) is over-determined and its compatibility conditions $x_{p\theta} = x_{\theta p}$ and $y_{p\theta} = y_{\theta p}$ lead to the following equations relating F and φ

$$F_\theta = \frac{p}{u} \varphi_p, \quad F_p = p \frac{\partial}{\partial p} \left(\frac{1}{up} \right) \varphi_\theta. \quad (10)$$

The system of Eqs. (10) is also over-determined and it is compatible if and only if $F_{p\theta} = F_{\theta p}$, i.e.

$$\frac{\partial}{\partial p} \left(\frac{p}{u} \varphi_p \right) = p \frac{\partial}{\partial p} \left(\frac{1}{up} \right) \varphi_{\theta\theta}. \quad (11)$$

Introducing the variable $\sigma = \int_0^p u/p' dp'$, Eq. (11) reduces to the following *generalized Tricomi* equation

$$\varphi_{\sigma\sigma} + K(\sigma) \varphi_{\theta\theta} = 0, \quad (12)$$

where $K(\sigma) = -p \frac{\partial}{\partial \sigma} \left(\frac{1}{up} \right)$. In the derivation of Eq. (12), it is crucial that $u = u(\eta) = u(p^2/2)$ does not depend on the variable θ .

4 Solution for a Kerr-type nonlinear response

A focussing Kerr-type medium is specified by the intensity law of the form $u = c_0(2\eta)^\gamma$, where c_0 and γ are certain positive constants. In this case, we

have $\sigma = c_0 p^{2\gamma}/(2\gamma)$ and $u = c_0 p^{2\gamma}$. Hence, the generalized Tricomi equation reduces to the elliptic equation of the form

$$\varphi_{\sigma\sigma} + \frac{\tilde{\gamma}}{\sigma^2} \varphi_{\theta\theta} = 0, \quad (13)$$

where $\tilde{\gamma} = (2\gamma + 1) / (4\gamma^2)$. It is *a priori* not obvious that the exponent of σ in Eq. (13) does not depend on γ .

Equation (13) admits the following separable variables solution

$$\varphi = \left(\frac{c_0}{2\gamma} \right)^\alpha \cos \lambda \theta p^{2\gamma\alpha} \quad (14)$$

where $\alpha = (1 + \sqrt{1 + 4\tilde{\gamma}\lambda^2})/2$ is a positive real parameter and λ is the separation of variables constant and plays the role of a geometric deformation parameter. Integration of Eqs. (10) gives

$$F = \alpha \left(\frac{c_0}{2\gamma} \right)^{\frac{\tilde{\alpha}}{2\gamma}} \frac{\sin \lambda \theta}{\lambda} p^{\tilde{\alpha}}, \quad (15)$$

where $\tilde{\alpha} = 2\gamma(\alpha - 1)$.

Assuming $\lambda^2 \neq 1$ and integrating Eqs. (9), we obtain

$$x = \left(\frac{c_0}{2\gamma} \right)^{\frac{\tilde{\alpha}}{2\gamma}} A(\theta) p^\beta, \quad y = \left(\frac{c_0}{2\gamma} \right)^{\frac{\tilde{\alpha}}{2\gamma}} B(\theta) p^\beta, \quad (16)$$

where

$$A(\theta) = -\frac{1}{\beta} \left[\alpha \sin \theta \cos \lambda \theta - \frac{1 + 2\gamma}{2\gamma} \lambda \cos \theta \sin \lambda \theta \right]$$

$$B(\theta) = \frac{1}{\beta} \left[\alpha \cos \theta \cos \lambda \theta + \frac{1 + 2\gamma}{2\gamma} \lambda \sin \theta \sin \lambda \theta \right],$$

and $\beta = 2\gamma(\alpha - 1) - 1$.

Let us assume that $|\lambda| < 1$. In this case, it is $\beta < 0$ and the formula (16) implies that $p \rightarrow \infty$ as $(x, y) \rightarrow (0, 0)$. Consequently, the intensity $u = c_0 p^{2\gamma}$ is divergent at the origin. From the formula (15) it follows that the phase $S = z + F(x, y)$ is also divergent at the origin if $\lambda \neq 0$.

We point out that the solution (15) is a deformation of the SV. Indeed, we have

$$\lim_{\lambda \rightarrow 0} F = \theta = \arctan(x/y),$$

On the hodograph plane, the unit vector $\mathbf{n} = \text{grad } S / |\text{grad } S|$ normal to the wave front takes the following remarkably simple form

$$\left(\frac{p}{\sqrt{1+p^2}} \cos \theta, \frac{p}{\sqrt{1+p^2}} \sin \theta, \frac{1}{\sqrt{1+p^2}} \right). \quad (17)$$

Near the origin (i.e. $p \rightarrow \infty$), we have $\mathbf{n} \sim (\cos \theta, \sin \theta, 0)$. Then, the unit vector normal to the wave front is completely undetermined. Nevertheless, at the point $(x, y) = (0, 0)$ the phase (15) is undetermined only for $\lambda = 0$. We call *hodographic vortex* a solution possessing a point of divergent phase where the unit vector normal to the wave-front is completely undetermined. From a physical point of view, approaching the origin, the electric field develops very strong oscillations even for very small λ s. Moreover, since the intensity diverges at the origin, the z -axis is a caustics. Despite the linear regime, where caustics and dislocation lines are complementary effects [5], in nonlinear geometric optics, vortices generate caustics. Approaching caustics, geometric optics approximation fails and the wave corrections become important.

Note also that the phase associated with hodographic vortices depends on the nonlinearity strength γ . This means that hodographic vortices are sensitive to the form of the nonlinearity.

5 Conclusions

Hodographic vortices arise in nonlinear geometric optics as deformation of SVs. Unlike SVs, the refractive index distribution induced by a hodographic vortex is sensitive to the form of the nonlinearity.

Intriguing features of hodographic vortices stimulate the investigation of further significant solutions to the equation (13). Moreover, the study of other physically relevant nonlinear responses will also be of interest.

We finally stress that screw type vortex solutions studied above arise from the analysis of a particular (stationary) reduction of the dispersionless NLS type equation (3). In general, one can construct an infinite family of reductions that are integrable via the hodograph method. Edge and edge-screw type phase dislocations are expected to be associated with such non stationary reductions.

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